Department of Economics **University of Manitoba**

ECON 7010: Econometrics I

Matrices: Concepts, Definitions & Some Basic Results

---

1. Concepts and Definitions

Vector

A "vector" is a set of scalar values, or "elements", placed in a particular order, and then displayed either as a column of values, or a row of values. The number of elements in the vector gives us the vector's "dimension".

So, the vector $v_1 = \begin{pmatrix} 2 & 6 & 3 & 8 \end{pmatrix}$ is a row vector with 4 elements – it is a (1×4) vector, because it has 1 row with 4 elements. We can also think of these elements as being located in "column" positions, so the vector essentially has one row and 4 columns.

Similarly, the vector $v_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \\ 2 \end{pmatrix}$ is a column vector with 4 elements – it is a (4×1) vector, because it has 1 column with 4 elements. We can think of these elements as being located in "row" positions, so the vector essentially has one column and 4 rows.

Matrix

A "matrix" is rectangular array of values, or "elements", obtained by taking several column vectors (of the same dimension) and placing them side-by-side in a specific order. Alternatively, we can think of a matrix as being formed by taking several row vectors (of the same dimension) and placing them one above the other, in a particular order.

For example, if we take the vectors $v_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \\ 2 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 6 \\ 2 \\ 7 \end{pmatrix}$ we can form the matrix

$V_1 = \begin{bmatrix} 2 & 1 \\ 5 & 6 \\ 8 & 2 \\ 2 & 7 \end{bmatrix}$. If we place the vectors side-by-side in the opposite order, we get a different matrix, of course, namely:

$$V_2 = \begin{bmatrix} 1 & 2 \\ 6 & 5 \\ 2 & 8 \\ 7 & 2 \end{bmatrix}.$$

Dimension of a Matrix

The "dimension" of a matrix is the number of rows and the number of columns. If there are "m" rows and "n" columns, the dimension of the matrix is ($m \times n$). You can see how the way in which the dimension of a vector was defined above is just a special case of this concept.

For example, the matrix $A = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 4 & 3 \\ 5 & 8 & 9 \end{bmatrix}$ is a (3×3) matrix, while the dimension of the matrix $D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$ is (3×2).

Square Matrix

A matrix is "square" if it has the same number of rows as columns.

The matrix $A = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 4 & 3 \\ 5 & 8 & 9 \end{bmatrix}$ is square, as it has 3 rows and 3 columns. The matrices

$D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 8 \\ 8 & 2 & 9 \end{bmatrix}$ are *not* square – they are "rectangular".

Rectangular Matrix

A rectangular matrix is one whose number of columns is different from its number of rows.

The matrices $D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 8 \\ 8 & 2 & 9 \end{bmatrix}$ are "rectangular". The matrix D has 3 rows and 2 columns – it is (3×2). The matrix E has 2 rows and 3 columns – it is (2×3).

Leading Diagonal

If the matrix is square, the "leading diagonal" is the string of elements from the top left corner of the matrix to the bottom right corner.

If $A = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 4 & 3 \\ 5 & 8 & 9 \end{bmatrix}$, its leading diagonal contains the elements (7, 4, 9).

Diagonal Matrix

A square matrix is said to be "diagonal" if the only non-zero elements in the matrix occur along the leading diagonal.

The matrix $C = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is a diagonal matrix.

Scalar Matrix

A square matrix is said to be "scalar" if it is diagonal, and all of the elements of its leading diagonal are the same.

The matrix $B = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is "scalar", but the matrix $C = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is not.

Identity Matrix

An "identity" matrix is one which is scalar, with the value "1" for each element on the leading diagonal. (Because this matrix is scalar, it is also a square and diagonal matrix.)

The matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix. (We might also name it I_3 to indicate that it is a (3×3) identity matrix.)

An identity matrix serves the same purpose as the number "1" for scalars – if we pre-multiply or post-multiply a matrix by the identity matrix (of the right dimensions), the original matrix is unchanged.

So, if $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$, then $ID = D = DI$.

Null Matrix

A "null matrix" is one which has the value zero for all of its elements. The matrices

$Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ are both null matrices.

A null matrix serves the same purpose as the number "0" for scalars – if we pre-multiply or post-multiply a matrix by the identity matrix (of the right dimensions), the result is a null matrix.

So, if $Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$, then $ZD = N$. [Note that Z is (3×3), and D is (3×2), so ZD must be (3×2).]

Trace

The "trace" of a square matrix is the sum of the elements on its leading diagonal.

For example, if $A = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 4 & 3 \\ 5 & 8 & 9 \end{bmatrix}$, then trace($A$) = (7 + 4 + 9) = 20.

Transpose

The "transpose" of a matrix is obtained by exchanging all of the rows for all of the columns. That is, the first row becomes the first column; the second row becomes the second column; and so on.

If $D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$, then the transpose of D is $D' = \begin{bmatrix} 1 & 6 & 8 \\ \\ 8 & 5 & 9 \end{bmatrix}$. Sometimes we write D^T

rather than D' to denote the transpose of a matrix. Note that if the original matrix is an ($m \times n$) matrix, then its transpose will be an ($n \times m$).

Recall that a vector is just a special type of matrix – a matrix with either just one row, or just one column. So, when we transpose a row vector we just get a column vector with the elements in the same order; and when we transpose a column vector we just get a row vector, with the order of the elements unaltered.

For example, when we transpose the (1×4) row vector, $v_1 = \begin{pmatrix} 2 & 6 & 3 & 8 \end{pmatrix}$, we get a column vector which is (4×1):

$$v_1' = \begin{pmatrix} 2 \\ 6 \\ 3 \\ 8 \end{pmatrix}.$$

Symmetric Matrix

A square matrix is "symmetric" if it is equal to its own transpose – that is, transposing the rows and columns of the matrix leaves it unchanged. In other words, as we look at elements above and below the leading diagonal, we see the same values in corresponding positions – the (i, j)'th. element equals the (j , i)th. element, for all $i \neq j$.

For example, let $F = \begin{bmatrix} 1 & 5 & 6 \\ 5 & 2 & 4 \\ 6 & 4 & 9 \end{bmatrix}$. Here the (1 , 3) element and the (3 , 1) element are both 6, *etc*. Note that $F' = F$, so F is symmetric.

Linear Dependency

Two vectors (and hence two rows, or two columns of a matrix) are "linearly independent" if one vector *cannot* be written as a multiple of the other. So, for example, the vectors x_1 = (1 , 3 , 4 , 6) and x_2 = (5 , 4 , 1 , 8) are linearly independent, but the vectors x_3 = (1 , 2 , 4 , 8) and x_4 = (2 , 4 , 8, 16) are "linearly dependent", because $x_4 = 2x_3$.

More generally, a collection of (say) n vectors is linearly independent if no one of the vectors can be written as a linear combination (weighted sum) of the remaining (n - 1) vectors. Consider the vectors x_1 and x_2 above, together with the vector x_5 = (4 , 1 , -3 , 2). These three vectors are *not* linearly independent, because $x_5 = x_2 - x_1$.

Rank of a Matrix

The "rank" of a matrix is the (smaller of the) number of linearly independent rows or columns in the matrix.

For example, the matrix $D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$ has a rank of "2". It has 2 columns, and the first column is *not* a multiple of the second column. The columns are linearly independent. It has 3 rows – these three rows make up a group of 3 linearly independent vectors, but by convention we define "rank" in terms of the smaller of the number of rows and columns. So this matrix has "full rank".

On the other hand, the matrix $G = \begin{bmatrix} 1 & 5 & 6 \\ 5 & 2 & 7 \\ 6 & 4 & 10 \end{bmatrix}$ has a rank of "2", because the third column is the sum of the first two columns. In this case the matrix has "less than full rank", because potentially it could have had a rank of "3", but the one linear dependency reduces the rank below this potential value.

Determinant of a Matrix

The determinant of a (square) matrix is a particular polynomial in the elements of the matrix, and is a scalar quantity. We usually denote the determinant of a matrix A by $|A|$, or det.(A).

The determinant of a scalar is just the scalar itself.

The determinant of a (2×2) matrix is obtained as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22}) - (a_{21}a_{12}) .$$

If the matrix is (3×3), then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{11}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

which can then be expanded out completely, and we see that it is just a polynomial in the a_{ij} elements.

Principal Minor Matrices

Let A be an $(n \times n)$ matrix. Then the "principal minor matrices" of A are the sub-matrices formed by deleting the last $(n - 1)$ rows and columns (which leaves only first diagonal element); then deleting the last $(n - 2)$ rows and columns (which leaves the leading (2×2) block of A); then deleting the last $(n - 3)$ rows and columns; *etc*.

If $A = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 4 & 3 \\ 5 & 8 & 9 \end{bmatrix}$, its first principal minor matrix is $A_{(1)} = 7$; the second principal minor matrix is $A_{(2)} = \begin{bmatrix} 7 & 3 \\ 6 & 4 \end{bmatrix}$; and the third is just A itself.

Note: The term "principal minor" is often used as an abbreviation for "determinant of the principal minor matrix", so you need to be careful.

Inverse Matrix

Suppose that we have a square matrix, A. If we can find a matrix B, with the same dimension as A, such that $AB = BA = I$ (an identity matrix), then B is called the "inverse matrix" for A, and we denote it as $B = A^{-1}$**.**

Clearly, the inverse matrix corresponds to the reciprocal when we are dealing with scalar numbers. Note, however, that many square matrices *do not* have an inverse.

Singular Matrix

A square matrix that does *not* have an inverse is said to be a "singular matrix". On the other hand, if the inverse matrix *does* exist, the matrix is said to be "non-singular".

For example, every null matrix is singular. Similarly every identity matrix is non-singular, and equal to its own inverse (just as 1/1 = 1 in the case of scalars).

Computing an Inverse Matrix

You will not have to construct inverse matrices by hand, except in very simple cases – a computer can be used instead. It is worth knowing how to obtain the inverse of a (non-singular) matrix when the matrix is just (2 × 2). In this case we first obtain the determinant of the matrix. We then interchange the 2 elements on the leading diagonal of the matrix, and change the signs of the 2 off-diagonal elements. Finally, we divide this transformed matrix by the determinant. Of course, this can only be done if the determinant is non-zero! So, a necessary (but not sufficient) condition for a matrix to be non-singular is that its determinant is non-zero.

To illustrate these calculations, consider the matrix

$R = \begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}$. Its determinant is Δ = [(4)(-2) – (1)(-1)] = [-8 + 1} = -7. So, the inverse of

R is the matrix

$R^{-1} = \left(\frac{1}{\Delta}\right)\begin{bmatrix} -2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ 1/7 & -4/7 \end{bmatrix}$. You can check that $RR^{-1} = R^{-1}R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Definiteness of a Matrix

Suppose that A is any square ($n \times n$) matrix. The A is "positive definite" if the (scalar) quadratic form, $x'Ax > 0$, for *all* non-zero ($n \times 1$) vectors, x; A is "positive semi-definite" if the (scalar) quadratic form, $x'Ax \geq 0$, for *all* non-zero ($n \times 1$) vectors, x; A is "negative definite" if the (scalar) quadratic form, $x'Ax < 0$, for *all* non-zero ($n \times 1$) vectors, x; and A is "negative semi-definite" if the (scalar) quadratic form, $x'Ax \leq 0$, for *all* non-zero ($n \times 1$) vectors, x. If the sign of $x'Ax$ varies with the choice of x, then A is said to be "indefinite".

For example, let $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$x'Ax = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4x_1 \\ 2x_2 \end{pmatrix} = 4x_1^2 + 2x_2^2 > 0$, unless

both x_1 and x_2 are zero. So, A is positive definite in this case.

Idempotent Matrix

Suppose that we have a square and symmetric matrix, Q, which has the property that $Q^2 = Q$. Because Q is symmetric, this means that $Q'Q = QQ' = QQ = Q^2 = Q$. Any matrix with this property is called an "idempotent matrix".

Clearly, the identity matrix, and the null matrix are idempotent. This corresponds with the fact that the only two idempotent scalar numbers are unity and zero. However, other matrices can also be idempotent.

Let X be an $(T \times k)$ matrix, with $T > k$, and such that the square, $(k \times k)$ matrix $(X'X)$ has an inverse (*i.e.*, it is non-singular). Let $P = X(X'X)^{-1}X'$. Note that P is an $(T \times T)$ matrix, so it is square; and also note that

$$P' = [X(X'X)^{-1}X']' = (X')'[(X'X)^{-1}]'X' = X[(X'X)']^{-1}X' = X(X'X)^{-1}X' = P\,.$$

That is, P is symmetric. Now, observe that

$$\begin{aligned} P'P &= [X(X'X)^{-1}X']'X(X'X)^{-1}X' = X[(X'X)']^{-1}X'X(X'X)^{-1}X' = X\,I(X'X)^{-1}X' \\ &= X(X'X)^{-1}X' = P \end{aligned}$$

and so P is idempotent. You can also check that the matrix $M = (I_T - P)$ is another example of an idempotent matrix.

2. Some Basic Matrix Results

Let A be a square $(n \times n)$ matrix. Then:

1. Let X be an $(m \times n)$ matrix with full rank. Then (XAX') is positive definite if A is positive definite.

2. If A is non-singular (that is, it has an inverse) then it is either positive definite, or negative definite, and its determinant is non-zero.

3. If A is positive semi-definite or negative semi-definite, then its determinant is zero, and it is singular (it does not have an inverse).

4. *If* A is positive definite then the determinant of A is positive.

5. *If* A is positive (semi-) definite then all of the leading diagonal elements of A are positive (non-negative).

6. *If* A is negative (semi-) definite then all of the leading diagonal elements of A are negative (non-positive).

7. A is positive definite *if and only if* the determinants of all of its principal minor matrices are positive.

8. A is negative definite *if and only if* the determinants of the principal minor matrices of order k have sign $(-1)^k$, $k = 1, 2, ..., n$. (That is, - , +, - , +,...........)

9. Suppose that B is also $(n \times n)$, and that both A and B are non-singular. Then the definiteness of $(A - B)^{-1}$ is the same as the definiteness of $(B^{-1} - A^{-1})$.

10. If A is either positive definite or negative definite, then rank(A) = n.

11. If A is positive semi-definite or negative semi-definite, then rank(A) = $r < n$.

12. If A is idempotent then it is positive semi-definite.
13. If A is idempotent then rank(A) = trace(A), where the trace is the sum of the leading diagonal elements.

14. If C is an $(m \times n)$ matrix, then the rank of C cannot exceed min.(m , n).

15. If A is positive semi-definite or negative semi-definite, then rank(A) = $r < n$, and it has "r" non-zero eigenvalues

16. If A is either positive definite or negative definite then all of its eigenvalues are non-zero.

17. Suppose that A and B are both $(n \times n)$ matrices. Then trace($A + B$) = trace(A) + trace(B).

18. Suppose that A and B are both $(n \times n)$ matrices. Then $(A + B)' = (A' + B')$.

19. Suppose that A is a *non-singular* $(n \times n)$ matrix, then $(A^{-1})' = (A')^{-1}$.

20. Suppose that A and B have dimensions such that AB is defined. Then $(AB)' = (B'A')$.

21. Suppose that A and B are *non-singular* $(n \times n)$ matrices such that both AB and BA are defined. Then $(AB)^{-1} = (B^{-1}A^{-1})$.

22. If D is a square diagonal matrix which is non-singular, then D^{-1} is also diagonal, and the elements of the leading diagonal are the reciprocals of those on the diagonal of D itself.